Geogebra Assignment for 251

General Information:

- Geogebra is a free digital tool for mathematical learning. If you go to their website https://www.geogebra.org/, and click on "**App Downloads**" on the left column, you will notice that you can either download the app, or just click on "Start" if you want to use a particular feature of Geogebra online.
- The solutions for this assignment should consist of a document (for example a word or pdf file) where you include all the images you generated, plus any steps in the solution of the problem that were used for arriving at these images.

Part 1

Vector Projection:

To develop some intuition about vector projections, we will use the animation https://www.geogebra.org/m/mkV7F8Jf to find some of them visually.

Exercise 1. Consider the vectors $\mathbf{u} = \langle 2, 3\rangle$ and $\mathbf{v} = \langle 4, 2\rangle$.

a) Show these vectors together with $\mathrm{proj}_{\mathbf{v}}\mathbf{u}$.

b) Now do the same but for $2\mathbf{u}$, $\mathbf{v}$ and $\mathrm{proj}_{\mathbf{v}}2\mathbf{u}$. How does it compare to $\mathrm{proj}_{\mathbf{v}}\mathbf{u}$?

c) Now do the same for $\mathbf{u}$, $-\frac{1}{2}\mathbf{v}$ and $\mathrm{proj}_{-\frac{1}{2}\mathbf{v}}\mathbf{u}$. How does it compare to $\mathrm{proj}_{\mathbf{v}}\mathbf{u}$?

d) More generally, what should be relation between $\mathrm{proj}_{\mathbf{v}}(a\mathbf{u})$ and $\mathrm{proj}_{\mathbf{v}}\mathbf{u}$, where a is a non-zero constant (rescaling factor)? How about the relation between $\mathrm{proj}_{\mathbf{v}}\mathbf{u}$ and $\mathrm{proj}_{a\mathbf{v}}\mathbf{u}$, where again a is a non-zero constant?

Parametrizing Curves:

Viviani's curve is defined as the intersection of the sphere $x^2 + y^2 + z^2 = 4$ with the cylinder $(x-1)^2 + y^2 = 1$. Observe that

$$\begin{cases} x = 1 + \cos t \\ y = \sin t \\ z = \pm 2 \sin\left(\frac{t}{2}\right) \end{cases}$$

gives a parametrization of this curve [this uses the identity $\sin\left(\frac{t}{2}\right) = \pm\sqrt{\frac{1-\cos t}{2}}$].

To plot Viviani's curve go to "**3D Calculator**" and type "curve": choose the option for three entries

Curve(Expression, Expression, Expression, Parameter Variable, Start Value, End Value)

and enter the formulas for x, y, z in terms of t here (choose the positive sign for z and t from 0 to π, which in Geogebra you can write as "pi". You must also write sin(t) instead of sint for example)

Exercise 2. Save the image you get as a pdf or jpeg file.

Exercise 3. Now plot the curve which is obtained as the intersection of the paraboloid $z = x^2 + y^2$ with the cylinder $(x-1)^2 + y^2 = 1$. Use the interval [0 2*pi] instead of [0 pi]. Hint: notice that this is the same cylinder which appeared in Viviani's curve so x and y are given by the same functions of t. Only z requires a different formula in terms of t.

Graphing Functions:

Consider the function

$$f(x, y) = y\left(1 - \frac{1}{(x^2 + y^2)}\right)$$

Exercise 4. What is the domain of this function?

To plot it go to 3D calculator of Geogebra and type f(x,y)= y (1- 1/(x^2+y^2)).

Exercise 5. Save the image you get as a pdf or jpeg file. Notice that when you plot the graph of this function, away from the origin it looks very similar to a plane, can you think of why?

To graph level curves go to https://www.geogebra.org/ and enter "Level Curves" on the search bar. There is a nice animation created by Kristen Beck (https://www.geogebra.org/m/J3kDCzjz). Use this to see some level curves of this function.

Exercise 6. Save the image you get as a pdf or jpeg file (take a screenshot of the level curves shown on the xy plane)

Part 2

Partial Derivatives and the Gradient

To plot 2d vector fields on Geogebra enter "two dimensional vector field" on the Geogebra.com search bar. There is an useful one with the link here https://www.geogebra.org/m/kdw2vf9p.

Exercise 7. Find the gradient of $h(x, y) = \frac{1}{2}(x^2 + y^2)$ and show the image you get.

For 3d vector fields go to the Geogebra.com search bar and enter "vector fields". An useful one is this animation https://www.geogebra.org/m/u3xregNW.

Exercise 8. Plot the vector field $\mathbf{F} = (x + y)\mathbf{i} + (z - y)\mathbf{j} + (x + y + z)\mathbf{k}$ and show the image you get.

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Exercise 9. Use the 3D calculator option to plot the ellipsoid $f(x, y, z) = x^2 + 2y^2 + 3z^2 = 1$ together with the tangent plane at the point $(1, 0, 0)$.

Lagrange Multipliers

In Calculus 1 you learned how to solve the following optimization problem:

"Suppose that you have two numbers x, y and want to make $x^2 + y^2$ as large or small as possible, subject to the condition that the point (x, y) must belong the parabola $x = y^2 - 5$. "

Exercise 10. a) Use Calculus 1 techniques to find a function $S(x)$ you need to optimize and find the value(s) of x that works, together with the corresponding value(s) of y.

b) Now we will reinterpret this problem as a Lagrange multiplier problem. This means that we have a function of two variables $f(x, y) = x^2 + y^2$ and a constraint condition $g(x, y) = y^2 - x = 5$. Find the values of x, y that work using the Lagrange Multiplier method. Are these the same points as those of part a) ?

c) To understand what could be going on, suppose we had solved part a) differently. Namely, that someone writes the function to optimize in terms of y, not x. Do this using Calculus 1 techniques and see what values of x, y you get.

d) Use the animation https://www.geogebra.org/m/PSzG4pe6 to show that at these three points the gradients ∇f and ∇g are parallel. Generate an image for each point. In the animation the blue vector represents $\pm\nabla g$ and the red vector represents $\pm\nabla f$.

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Part 3

Geogebra also allows you to visualize the Jacobian in useful ways. For example, if you enter "Jacobian Animation" on the Geogebra.com search bar, you will find the following animation https://www.geogebra.org/m/HpH5NX7U.

Exercise 11. Enter the transformation $x = u - v$, $y = u + v$. What does the rectangle $0 \le u \le 2$, $-1 \le v \le 1$ become on the xy plane? Use the button "animation" and show the uv picture, as well as the xy picture. What is the area of the figure on the xy plane? What is the Jacobian factor? Can you see why the Jacobian is needed in order to correct for the distortion introduced when switching to the uv plane?

Surfaces

Parametrizing Surfaces:

Exercise 12. Parametrize the monkey saddle $z = x^3 - 3xy^2$ by using $u = x$, $v = y$ as the parameters. On the 3D calculator option of Geogebra use the command

"Surface(Expression, Expression, Expression, Parameter Variable 1, Start Value, End Value, Parameter Variable 2, Start Value, End Value)".

Take the values for u, v between -1 and 1.

Divergence and Curl:

For a vector field $\mathbf{F}(x, y) = \langle F_1(x, y), F_2(x, y), 0\rangle = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$ in the plane, the divergence is given by

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

while the curl is given by

$$\nabla \times \mathbf{F} = \left\langle 0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

Since the first two entries in this case will always be zero, we can just focus on the third entry, which is sometimes denoted $\text{curl}_z\mathbf{F}$.

Exercise 13. Find $\nabla \cdot \mathbf{F}$ and $\text{curl}_z\mathbf{F}$ for:

a) $\mathbf{F} = \langle x, y, 0\rangle$

b) $\mathbf{F} = \langle -y, x, 0\rangle$

c) $\mathbf{F} = \left\langle 2xy, y^2 \right\rangle$

d) Use the animation on https://www.geogebra.org/m/XfmAAUTG to show the pictures you get for the three previous vector fields. Notice that curl_z can be interpreted as the angular velocity of a paddle wheel you place at (x, y), if you think of the vector field as giving you the velocity of a fluid.